Thermal instability in a horizontal fluid layer: effect of boundary conditions and non-linear temperature profile

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(Received 7 August 1963 and in revised form 28 October 1963)

An investigation is carried out to determine the conditions marking the onset of convective motion in a horizontal fluid layer in which a negative temperature gradient occurs somewhere within the layer. In such cases, fluid of greater density is situated above fluid of lesser density. Consideration is given to a variety of thermal and hydrodynamic boundary conditions at the surfaces which bound the fluid layer. The thermal conditions include fixed temperature and fixed heat flux at the lower bounding surface, and a general convective–radiative exchange at the upper surface which includes fixed temperature and fixed heat flux as special cases. The hydrodynamic boundary conditions include both rigid and free upper surfaces with a rigid lower bounding surface. It is found that the Rayleigh number marking the onset of motion is greatest for the boundary condition of fixed temperature and decreases monotonically as the condition of fixed heat flux is approached. Non-linear temperature distributions in the fluid layer may result from internal heat generation. With increasing departures from the linear temperature profile, it is found that the fluid layer becomes more prone to instability, that is, the critical Rayleigh number decreases.

1. Introduction

When a horizontal layer of fluid is heated from below, the density of the fluid near the top of the layer is greater than that of the fluid adjacent to the heating surface. At first thought, it would appear that such an arrangement of the fluid would be unstable and immediately break down into a convective motion. However, it has been shown both theoretically and experimentally that the fluid can, in fact, remain in a quiescent state for a range of values of the temperature difference between the bottom and top surfaces. When this temperature difference exceeds a critical value which depends on the layer thickness and the fluid properties, then the quiescent state breaks down and thermal convection sets in.

The earliest experiments which called attention to the aforementioned thermal instability phenomenon are reported by Thomson (1882) and by Bénard

(1900). Of these, the latter investigator presented a much more complete description of the development of the convective flow. As a consequence of this work, the thermal instability problem is frequently referred to as the Bénard problem.† The first analytical treatment aimed at determining the conditions delineating the breakdown of the quiescent state was carried out by Lord Rayleigh (1916). Rayleigh's work was generalized and also extended to a broader range of boundary conditions by Jeffreys (1926, 1928), Low (1929), and Pellew & Southwell (1940). The latter of these presents the most complete theory of the thermal instability problem available up to this time. A very valuable survey of both theoretical and experimental investigations relating to thermal instability has recently been published by Ostrach (1957).

The aforementioned analytical studies have considered hydrodynamic boundary conditions which correspond to the following containment conditions of the fluid layer: (1) the upper and lower bounding surfaces are both rigid; (2) the lower surface is rigid, while the upper surface is free; (3) the upper and lower surfaces are both free. The latter condition does not appear to correspond to a real physical situation but may be of theoretical interest.

The thermal conditions usually applied at the upper and lower surfaces of the fluid are based on the supposition that these surfaces are in contact with materials of infinite thermal conductivity and heat capacity. From such a model, it follows that the temperatures at the surfaces are not perturbed when the quiescent state breaks down. In a special case, Jeffreys (1926) imposed a condition which he described as corresponding to an insulated upper surface. Objections to this condition have been raised both by Low and by Pellew & Southwell on the grounds that it precludes a steady non-zero temperature distribution in the fluid.

Consideration of actual physical situations suggests that the heretofore standard thermal boundary conditions of fixed temperatures at the surfaces of the fluid layer may be too restrictive. For example, when the upper surface is free, there will be a heat exchange between the free surface and the environment. If the heat-transfer coefficient between the surface and the environment is finite, the surface temperature will be perturbed when the quiescent state breaks down. One may cite various other illustrations of the restrictive nature of the fixed-temperature boundary conditions, but perhaps one other will suffice. For instance, if the heating at the lower surface is accomplished by passing an electric current through a thin metallic foil, then the boundary condition at the lower surface may be more nearly a fixed heat flux rather than a fixed temperature.

Considerations such as the foregoing provide the motivation for the first portion of the present investigation. The aim of this study is to determine analytically the conditions for the onset of convection for a broad range of thermal boundary conditions. Specifically, consideration is given to the case of arbitrary thermal conditions (i.e. arbitrary heat-transfer coefficient) at either

† It has been recently suggested by Block (1956) and by Pearson (1958) that the flow observed by Bénard was caused by surface tension rather than by gravity.

free or rigid upper surfaces corresponding to either fixed temperature or to fixed heat flux at a rigid lower surface.†

In prior analytical work on thermal instability, it has been customary to deal with a quiescent state characterized by a fluid temperature which is decreasing linearly with height. It is of interest to determine in what way the stability would be affected if the quiescent state were characterized by a non-linear temperature profile. Such a non-linear profile could arise if there were an internal heat generation within the fluid; for example, due to Joule heating in an electrolyte. Consequently, the second portion of this investigation is devoted to study of thermal instability in the presence of a non-linear temperature profile, such as that due to internal heat generation.

To begin the presentation, certain aspects of the analysis common to both portions of the investigation will be developed. The general solution will then be specialized to the specific studies relating to the thermal boundary conditions and to the fluid temperature profile.

2. General theory

Consideration is given to a horizontal fluid layer bounded above and below by surfaces through which heat may flow into or out of the fluid. In addition, there may be an internal heat generation within the fluid. The horizontal extension of the fluid layer is sufficiently great so that edge effects may be neglected. If the temperature within the layer were increasing monotonically upward under steady-state conditions, then lighter fluid would always be above heavier fluid and there would be static equilibrium without motion.‡ Under certain conditions, a quiescent steady state can also occur when heavier fluid lies above lighter fluid. The existence of such steady states can be investigated by finding the conditions under which a given density distribution (i.e. a given temperature distribution) is stable against small disturbances.

The thickness of the fluid layer will be denoted by L, with z measuring distances vertically upward; $z = 0$ corresponds to the lower surface of the layer and $z = L$ corresponds to the upper surface. The co-ordinate axes x and y lie in a horizontal plane. Under steady-state quiescent conditions, the temperature distribution in the fluid depends only on z

$$T_{ss} = -(S/2k)\,z^2 + Az + B, \tag{1}$$

in which S is a uniformly distributed internal heat source (energy/volume-time) and k is the thermal conductivity. The integration constants A and B may be evaluated from the thermal boundary conditions at $z = 0$ and $z = L$. In order to keep the analysis general, these boundary conditions will not be specified at this time.

† It has been pointed out to the authors that thermal conditions at the upper surface similar to those of the present investigation have also been considered in a recent doctoral dissertation by Sani (1963). Sani's work was concerned with the case in which both the upper and lower boundaries are free surfaces. However, it is just this case that was excluded from the present investigation.

‡ These considerations apply to fluids whose density decreases with temperature.

To investigate the conditions under which the quiescent solution is stable against small disturbances, one postulates a slightly perturbed state such that

$$T(x,y,z,t) = T_{ss}(z) + T^*(x,y,z,t), \quad p(x,y,z,t) = p_{ss}(z) + p^*(x,y,z,t), \tag{2a}$$

$$u = u^*(x,y,z,t), \quad v = v^*(x,y,z,t), \quad w = w^*(x,y,z,t), \tag{2b}$$

in which u, v and w are velocity components corresponding to x, y and z; t is the time, and p is the static pressure. The quantities T^*, u^*, v^* and w^* are taken to be sufficiently small for their squares and products to be neglected. The equations expressing conservation of mass, momentum and energy may now be written and the foregoing perturbations introduced. By carrying out manipulations similar to those of Pellew & Southwell, it is possible to eliminate all perturbation quantities except w^*, the governing equation for which is

$$\left(\frac{\partial}{\partial t} - \alpha\nabla^2\right)\left(\frac{\partial}{\partial t} - \nu\nabla^2\right)\nabla^2 w^* + g\beta\left(-\frac{S}{k}z + A\right)\nabla^2_{xy} w^* = 0, \tag{3}$$

in which α is the thermal diffusivity, β the coefficient of thermal expansion, ν the kinematic viscosity, g the acceleration of gravity, and ∇^2_{xy} the two-dimensional Laplace operator.

A separable solution for w^* may be sought in the form (Lin 1958)

$$w^* = F(z)\, G(x,y)\, e^{\sigma t}, \quad T^* = H(z)\, G(x,y)\, e^{\sigma t}. \tag{4}$$

From the equation of energy conservation, it can be shown that

$$\nabla^2_{xy} G + (a/L)^2 G = 0, \tag{5}$$

in which (a/L) is a constant arising from the separation of variables, wherein a is dimensionless. Further, if we substitute the proposed solution for w^* into (3) and utilize the condition (5) for G, we obtain

$$\frac{d^6F}{dZ^6} - 3a^2\frac{d^4F}{dZ^4} + 3a^4\frac{d^2F}{dZ^2} + (\Lambda + \Omega Z)\,F + \sigma^2\psi_1(Z) + \sigma\psi_2(Z) = 0, \tag{6}$$

where $$Z = z/L, \quad \Lambda = -(g\beta L^4 A a^2/\alpha\nu) - a^6, \quad \Omega = g\beta L^5 S a^2/\alpha\nu k. \tag{7}$$

The groups Λ and Ω represent constants which may be prescribed. The ψ_1 and ψ_2 functions need not be stated inasmuch as they drop out of the forthcoming analysis.

It has been shown by Pellew & Southwell that the threshold of instability is marked by $\sigma = 0$. Consequently, the last two terms of equation (6) may be deleted. The thus-reduced form of equation (6) is a homogeneous ordinary differential equation for the perturbation function F. As will be demonstrated later, the boundary conditions are also homogeneous. The resulting eigenvalue problem for this homogeneous system provides a means for determining the conditions under which a solution for the perturbation can exist.

2.1. *Solution of the perturbation equation*

A general solution of equation (6) with $\sigma = 0$ can be constructed in the form

$$F(Z) = \sum_{i=0}^{5} c_i f^{(i)}(Z), \tag{8}$$

with c_i arbitrary, in which the $f^{(i)}$ are rapidly convergent power series,

$$f^{(i)} = \sum_{n=0}^{\infty} b_n^{(i)} Z^n \quad (i = 0, 1, \ldots, 5). \tag{9}$$

The series coefficients $b_n^{(i)}$ obey the following recursion relationship for $n \geqslant 6$

$$b_n^{(i)} = \frac{1}{n!}\{3a^2(n-2)!\, b_{n-2}^{(i)} - 3a^4(n-4)!\, b_{n-4}^{(i)} - (\Lambda b_{n-6}^{(i)} + \Omega b_{n-7}^{(i)})\,(n-6)!\} \tag{9a}$$

and $b_{-1}^{(i)} = 0$. In addition, the $b_0^{(i)}$ through $b_5^{(i)}$ are specified as

$$b_n^{(i)} = \delta_{ni} \quad (0 \leqslant n \leqslant 5), \tag{9b}$$

wherein $\delta_{ni} = 1$ for $n = i$ and $\delta_{ni} = 0$ for $n \neq i$.

The constants $c_0, c_1, \ldots, c_5$ which appear in the solution for F are to be determined from the boundary conditions. A thorough-going investigation of various boundary conditions will be deferred until the next section of the paper. However, it is useful to specialize the solution (8) for a boundary condition that will be imposed on all of the various cases to be studied here. In practice, the lower bounding surface of the fluid layer will necessarily be a rigid surface (as opposed to a free surface). On such a surface, all the velocity components vanish identically (no slip), $u^* = v^* = w^* = 0$; correspondingly, $\partial u^*/\partial x = \partial v^*/\partial y = 0$. From the equation of continuity, it follows that $\partial w^*/\partial z = 0$. By applying the first of equations (4), the two aforementioned conditions on w^* can be restated in terms of the F function:

$$F = dF/dZ = 0, \quad \text{rigid surface.} \tag{10}$$

If the conditions for the rigid surface are applied at $z = 0$, it follows from equations (8) through (9b) that $c_0 = c_1 = 0$. Therefore, the solution for F corresponding to this physical situation becomes

$$F(Z) = \sum_{i=2}^{5} c_i f^{(i)}(Z). \tag{11}$$

2.2. *Solution for linear-temperature case*

The earlier literature has been concerned primarily with the case of temperature decreasing linearly with height, i.e. $S = 0$. It is of interest to apply the foregoing general solution to this case and, also, to give an alternative form of solution in terms of elementary functions that appears to have been previously overlooked.

If the temperatures of the lower and upper bounding surfaces are respectively T_1 and T_2 ($T_1 > T_2$), the integration constant A appearing in equation (1) becomes $(T_2 - T_1)/L$. With this A and with $S = 0$, the parameters Λ and Ω which enter the F equation take on the values

$$\Lambda = a^2\mathfrak{R} - a^6, \quad \Omega = 0 \tag{12}$$

wherein $\mathfrak{R}$ denotes the Rayleigh number

$$\mathfrak{R} = g\beta(T_1 - T_2)\, L^3/\alpha\nu. \tag{13}$$

When introduced into equation (9a), these Λ and Ω specialize the general solution for F to the case of the linearly-decreasing temperature distribution.

Alternatively, one may return to the differential equation (6) with these Λ and Ω and seek exponential solutions. After a lengthy calculation, it is found possible to construct the following solution for F which contains only real (i.e. non-imaginary) quantities:

$$F(Z) = d_0 \cosh r_1 Z \cos r_2 Z + d_1 \sinh r_1 Z \cos r_2 Z + d_2 \cosh r_1 Z \sin r_2 Z + d_3 \sinh r_1 Z \sin r_2 Z + d_4 \sin\{a(\chi-1)^{\frac{1}{2}} Z\} + d_5 \cos\{a(\chi-1)^{\frac{1}{2}} Z\}, \quad (14)$$

wherein
$$r_1 = \{(\omega_1^2+\omega_2^2)^{\frac{1}{2}}+\omega_1\}^{\frac{1}{2}}/\sqrt{2}, \quad r_2 = \{(\omega_1^2+\omega_2^2)^{\frac{1}{2}}-\omega_1\}^{\frac{1}{2}}/\sqrt{2}, \quad (14a)$$

and
$$\chi = (\Re/a^4)^{\frac{1}{3}}, \quad \omega_1 = a^2(1+\tfrac{1}{2}\chi), \quad \omega_2 = \tfrac{1}{2}a^2\chi\sqrt{3}. \quad (14b)$$

The d_0, d_5 are integration constants which remain to be determined from the boundary conditions.

At first glance, the solution (14) with its elementary functions may appear to be computationally more useful than the series solution (8). As will be seen later, however, the boundary conditions involve higher derivatives of the F function (up to the fifth). The fifth derivative of equation (14) is exceedingly lengthy, while the fifth derivative of the series solution (8) is relatively simple. From the numerical standpoint, the series solution has proved to be much more useful than the alternative solution (14).

With the general formulation of the problem thus complete, specific study may now be made of the effects of various boundary conditions and of a non-linear steady-state temperature distribution.

3. Stability criteria for various boundary conditions

3.1. *The hydrodynamic and thermal boundary conditions*

Consideration will be given here to determining how the stability of an initial quiescent state depends upon the thermal and hydrodynamic boundary conditions. For this phase of the study, the temperature distribution of the quiescent state will be taken as linearly decreasing with height (no internal heat sources or sinks, $S = 0$). The temperatures of the lower and upper bounding surfaces are respectively T_1 and T_2, with $T_1 > T_2$. For this situation, the solutions for the perturbation function F have already been derived and discussed in the preceding section of the paper.

As previously mentioned, the practical aspects of containment require that the lower bounding surface of the layer be rigid, and the corresponding conditions for the perturbation function F are stated in equation (10). On the other hand, the upper bounding surface may either be a free surface or a rigid surface. Practically speaking, if the fluid layer were a gas, the upper surface would in all likelihood be rigid; while if the layer were liquid, then the upper surface could equally well be free or rigid.

At a horizontal free surface, it is usually assumed that the vertical velocity component w^* vanishes; in addition, the free surface is not able to support a tangential shear so that $\partial u^*/\partial z = \partial v^*/\partial z = 0$. By applying the continuity equation, it follows from these later conditions that $\partial^2 w^*/\partial z^2 = 0$. The two aforementioned conditions on w^* can be rephrased in terms of the disturbance function F as

$$F = d^2F/dZ^2 = 0, \quad \text{free surface.} \quad (15)$$

Attention will now be directed to the thermal boundary conditions. If the wall which bounds the fluid layer has high heat conductivity and large heat capacity, then its temperature would be spacially uniform and unchanging in time. In other words, the boundary temperature would be unperturbed by any flow or temperature perturbations in the fluid. Thus

$$T^* = 0, \quad \text{fixed surface temperature.} \tag{16}$$

On the other hand, suppose that there is a uniform heat flux to the surface, which, for example, may be due to Joule heating in a bounding wall of appropriate thermal properties. According to Fourier's Law, the heat flux q passing through the boundary per unit time and area is

$$q = -k\,\partial T/\partial z, \tag{17}$$

in which k is the thermal conductivity of the fluid and $\partial T/\partial z$ is the temperature gradient in the fluid at the boundary. If q is unperturbed by thermal or flow perturbations in the fluid, it follows from the derivative of the first of equations (2a) that

$$\partial T^*/\partial Z = 0, \quad \text{fixed surface heat flux.} \tag{18}$$

If heat is being transferred at a free surface, then the energy conducted up to the surface, equation (17), must be carried away by convective–conductive (or perhaps radiative) transport to the environment. Such a transport is usually expressed in terms of the product of a heat transfer coefficient h and a temperature difference which is the driving force for heat transfer. A heat balance at the surface yields

$$-k\frac{\partial T}{\partial Z} = h(T - T_\infty), \tag{19}$$

where T_∞ is the temperature in the bulk of the environment. If one replaces T by $(T_{ss} + T^*)$ and notes that $-k(\partial T_{ss}/\partial z) = h(T_{ss} - T_\infty)$, there follows

$$\partial T^*/\partial Z = (hL/k)\,T^*, \quad \text{surface convection or radiation.} \tag{20}$$

The group (hL/k) is sometimes referred to as the Biot number. In addition to the free surface, the foregoing boundary condition also applies if the fluid layer is separated from an environment by a thin rigid lamina of negligible heat capacity. It is interesting to notice that for very small values of (hL/k), the boundary condition (20) approaches that for fixed heat flux, equation (18). On the other hand, when (hL/k) is very large, the condition (20) approaches that for fixed surface temperature, equation (16). In the development which follows, it will be assumed that h is invariant with time.

The thermal boundary conditions (16), (18), and (20) may be restated in terms of the disturbance variable F. From this it follows:

fixed surface temperature

$$\frac{d^4F}{dZ^4} - 2a^2\frac{d^2F}{dZ^2} + a^4F = 0; \tag{16a}$$

fixed surface heat flux

$$\frac{d^5F}{dZ^5} - 2a^2\frac{d^3F}{dZ^3} + a^4\frac{dF}{dZ} = 0; \tag{18a}$$

surface convection or radiation

$$\frac{d^5F}{dZ^5} - 2a^2\frac{d^3F}{dZ^3} + a^4\frac{dF}{dZ} = \frac{hL}{k}\left(\frac{d^4F}{dZ^4} - 2a^2\frac{d^2F}{dZ^2} + a^4F\right). \tag{20a}$$

The physical situations whose stability will be investigated in this paper may now be discussed. For all cases, the lower bounding surface will be regarded as rigid. Two thermal boundary conditions at the lower surface will be considered: fixed temperature and fixed heat flux. For each of these, the thermal boundary conditions imposed at the upper surface will be a general convective or radiative exchange with the environment as expressed by equations (20) or (20a). This includes as special cases the conditions of fixed temperature or fixed heat flux at the upper surface. Furthermore, separate consideration will be given to rigid and to free upper surfaces.

3.2. *Application of the boundary conditions*

To illustrate the manner in which the stability investigation is carried out, a detailed discussion will be given of the solution for one of the physical situations described above. The other physical situations are treated in a similar manner, with only small differences in detail.

Consider for concreteness the case of fixed temperature at the lower surface with a general convective or radiative transfer at an upper free surface. The solution to be applied is equation (11), wherein the $f^{(i)}$ are given by (9), (9a) and (9b). In turn, the Λ and Ω appearing in (7) are stated by equation (12). The boundary conditions appropriate to the particular case under consideration reduce to

$$Z = 0: \qquad \frac{d^4F}{dZ^4} - 2a^2\frac{d^2F}{dZ^2} = 0, \tag{21a}$$

$$Z = 1: \quad \left\{\begin{aligned} &F = \frac{d^2F}{dZ^2} = 0, \\ &\frac{d^5F}{dZ^5} - 2a^2\frac{d^3F}{dZ^3} + a^4\frac{dF}{dZ} = \frac{hL}{k}\frac{d^4F}{dZ^4}. \end{aligned}\right\} \tag{21b}$$

Upon applying these to equation (11), there follows

$$-4a^2c_2 + 24c_4 = 0, \tag{22a}$$

$$[c_2 f^{(2)} + c_3 f^{(3)} + c_4 f^{(4)} + c_5 f^{(5)}]_{Z=1} = 0, \tag{22b}$$

$$\left[c_2\frac{d^2f^{(2)}}{dZ^2} + c_3\frac{d^2f^{(3)}}{dZ^2} + c_4\frac{d^2f^{(4)}}{dZ^2} + c_5\frac{d^2f^{(5)}}{dZ^2}\right]_{Z=1} = 0, \tag{22c}$$

$$\sum_{i=2}^{5} c_i\left[\frac{d^5f^{(i)}}{dZ^5} - 2a^2\frac{d^3f^{(i)}}{dZ^3} + a^4\frac{df^{(i)}}{dZ} - \frac{hL}{k}\frac{d^4f^{(i)}}{dZ^4}\right]_{Z=1} = 0. \tag{22d}$$

The foregoing constitute four linear, homogeneous algebraic equations for the four constants, c_2, c_3, c_4 and c_5. Solution of such an algebraic system is possible only if the determinant of the coefficients of the c_i vanishes. The value of the determinant depends upon three parameters: the Biot number hL/k, the Rayleigh number $g\beta(T_1-T_2)L^3/\alpha\nu$, and the constant a which arose from the separation of variables. Suppose that the Biot number is held fixed. Then, for each and every a value which one might select, there can be found a Rayleigh number that causes the determinant of the coefficients to be zero. Moreover, it is found that for a particular a, the corresponding Rayleigh number has a value which is smaller than that for any other a. In other words, for every Biot number, there is a minimum Rayleigh number which permits a solution of the disturbance equation. Below this Rayleigh number, a solution for the disturbance equation cannot be found and this implies that the quiescent state is stable. Therefore, the aforementioned minimum Rayleigh number corresponds to the onset of instability. This is generally called the critical Rayleigh number.

The numerical computations which are required to find the critical Rayleigh number were carried out with the aid of a Control Data 1604 digital computer at the Numerical Analysis Center of the University of Minnesota. This machine operates with 10–11 decimal places (48 binary bits) and it therefore is able to yield results of high accuracy.

3.3. *Critical Rayleigh numbers*

The Rayleigh numbers marking the onset of instability are presented graphically in figures 1 (a) and (b). The first of these provides information for the case in which the lower surface is at a fixed temperature, while the second of these is for the case in which the lower surface is at a fixed heat flux. On each one of the figures, there are two curves: the dashed curve corresponds to a free upper surface and is referred to the left-hand ordinate scale; the solid curve corresponds to a rigid upper surface and is referred to the right-hand ordinate scale. The horizontal lines adjacent to the extremities of the curves provide the Rayleigh numbers for the limiting cases of $hL/k \to 0$ and $hL/k \to \infty$.

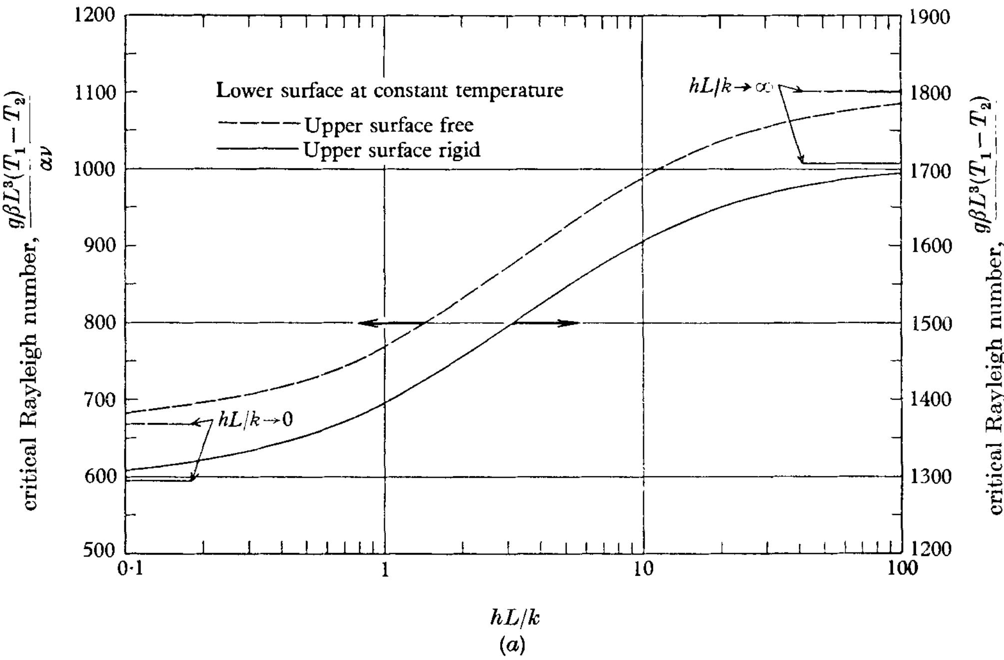
A parallel presentation of the results is made in tables 1 (a) and (b), wherein are also listed the a values corresponding to the critical Rayleigh numbers. The Rayleigh numbers tabulated therein are believed to be accurate to the number of significant figures shown.† The cases designated by $hL/k = 0$ and $hL/k = \infty$ respectively correspond to fixed heat flux and to fixed temperature at the upper surface.

Attention may first be directed to the results of figure 1 (a). From an inspection of the figure, it is seen that for a given hydrodynamic condition at the upper surface, the critical Rayleigh number‡ decreases monotonically with decreasing Biot number hL/k. Thus, the most stable situation corresponds to a fixed surface temperature. This agrees with an intuitive feeling that the fixing of the surface temperature should provide a stronger constraint against perturbation of the temperature profile than does the fixing of the temperature derivative at the surface. The critical Rayleigh number is most sensitive to Biot number in the mid range of Biot numbers. The foregoing remarks apply regardless of whether the upper surface is rigid or free. However, there is a marked difference in the numerical values of the critical Rayleigh numbers for these two cases. Generally speaking, the critical Rayleigh numbers for the rigid surface exceed those for the

† In general, the Rayleigh number has a very flat minimum as a function of a.

‡ Note that the Rayleigh number is based on the temperature difference $(T_1 - T_2)$, not on $(T_1 - T_\infty)$.

free surface by about 600, this difference being nearly uniform over the entire range of hL/k. That a fluid layer bounded by a rigid upper surface should be more stable is physically quite reasonable.



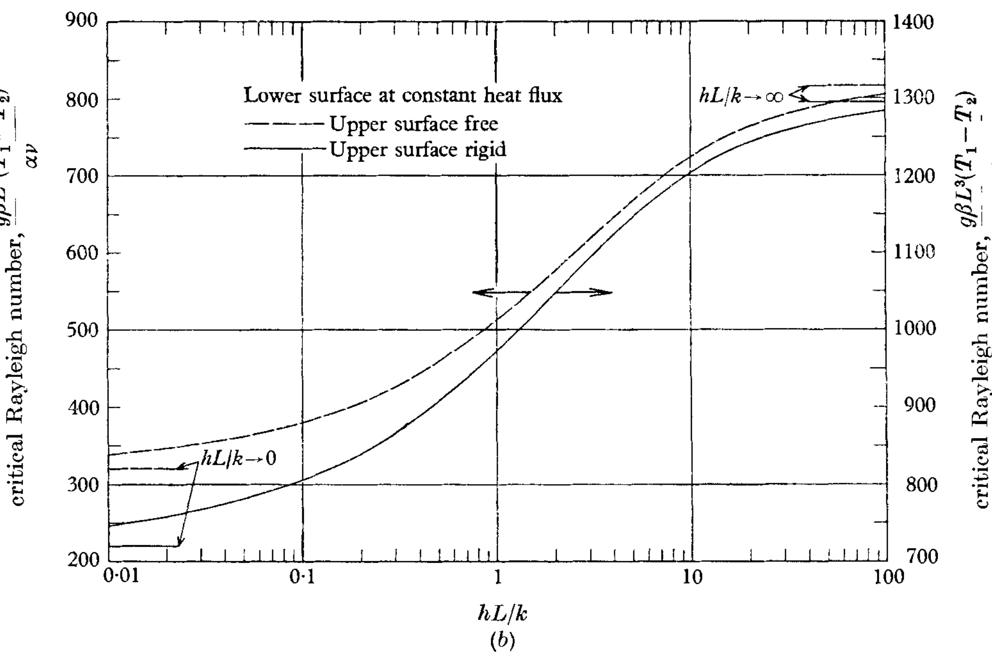


FIGURE 1. Critical Rayleigh numbers corresponding to linear initial temperature profile. (a) Fixed temperature at lower bounding surface; (b) fixed heat flux at lower bounding surface.

Consideration may next be given to the results of figure 1 (*b*) which correspond to the case of a fixed heat flux at the lower surface. From the figure, one sees that the critical Rayleigh number decreases with decreasing Biot number. So, as before, the most stable situation occurs when the temperature of the upper surface is fixed. Also, the quiescent state is more stable when the upper surface is rigid rather than free, the difference in the critical Rayleigh numbers between the two cases being on the order of 400.

	Upper surface free		Upper surface rigid	
hL/k	a	$\mathfrak{R}$	a	$\mathfrak{R}$
	(*a*) Lower surface at fixed temperature			
0†	2·09	669·001	2·55	1295·781
0·1	2·115	682·361	2·58	1309·545
0·3	2·17	706·365	2·64	1334·149
1	2·30	770·569	2·75	1398·508
3	2·46	872·506	2·90	1497·594
10	2·59	989·493	3·03	1607·104
30	2·65	1055·345	3·08	1667·102
100	2·67	1085·893	3·11	1694·573
∞‡	2·68	1100·657	3·12	1707·765
	(*b*) Lower surface at fixed heat flux			
0†	0	320·000	0	720·000
0·01	0·58	338·905	0·71	747·765
0·03	0·76	353·176	0·93	768·153
0·1	1·015	381·665	1·23	807·676
0·3	1·30	428·290	1·57	869·231
1	1·64	513·792	1·94	974·173
3	1·92	619·666	2·24	1093·744
10	2·11	725·150	2·44	1204·571
30	2·18	780·240	2·51	1259·884
100	2·20	804·973	2·53	1284·263
∞‡	2·21	816·748	2·55	1295·781

† Corresponds to fixed surface heat flux.
‡ Corresponds to fixed surface temperature.

TABLE 1. Critical Rayleigh numbers for linear temperature distribution.

There are some interesting differences in detail between the results of figures 1(*a*) and (*b*). First of all, by taking note of the ordinate scales, one sees that the Rayleigh numbers appearing on the former are higher. This is in accord with the previous finding that the case of fixed temperature is the most stable. Further comparison between the figures reveals that the asymptote at vanishing hL/k is approached more slowly in figure 1 (*b*) than in figure 1 (*a*).

It is of interest to inquire how the present results compare with those of other investigations. Only two of the entries in table 1 can be specifically compared. For the case of prescribed temperature at both boundaries and a rigid upper surface, Pellew & Southwell find a critical Rayleigh number of 1707·8, while that of the present investigation is 1707·765. For this same thermal condition, but with a free upper surface, the corresponding values are 1100·65 and 1100·657. The agreement is thus seen to be excellent. In addition, if figures 1 (*a*) and (*b*) were to be replotted on linear co-ordinates rather than on semi-logarithmic co-ordinates, the curves would have the same qualitative shape as that found by Sani for the case of two free surfaces.

As a final note, it may be worth while to indicate a typical value for hL/k. For a $\frac{1}{4}$-inch thick layer of liquid water, a reasonable value for the heat-transfer coefficient at its free surface might be 2 B.Th.U./hr ft.^{2} °F; then hL/k would be on the order of 0·1. If the lower surface had a fixed temperature, the critical Rayleigh number would be 1309. This is substantially less than the value 1708 which corresponds to a fixed temperature at the upper surface.

4. Stability criteria for non-linear fluid-temperature distributions

Consideration will now be given to investigating how the stability of a quiescent steady state is affected by the shape of the temperature distribution in the fluid. As has already been noted, prior analytical work has been generally concerned with an initial quiescent state in which the fluid temperature decreases linearly with height.† In the present study, the non-linearity in the temperature distribution is created by a uniformly distributed heat source S (e.g. Joule heating in an electrolyte). The boundary conditions selected for this phase of the investigation are rigid, fixed-temperature surfaces above and below.

4.1. *The temperature distributions*

If the temperatures of the lower and upper bounding surfaces are designated as T_1 and T_2, respectively, then the steady-state temperature distribution as given by equation (1) may be written as

$$(T-T_2)/(T_1-T_2) = 1-Z+N_s(Z-Z^2) \tag{23a}$$

or alternatively

$$(T-T_1)/(T_2-T_1) = 1-Z'+(-N_s)(Z'-Z'^2), \quad Z' = 1-Z \tag{23b}$$

in which N_s is a dimensionless grouping which is defined as

$$N_s = SL^2/2k(T_1-T_2). \tag{24}$$

Inasmuch as $(1-Z)$ represents the standard linearly decreasing temperature distribution, then the departure of N_s from zero is a measure of the non-linearity introduced by the heat source. As it is conceived of here, the heat source S will always be a positive number. Therefore, $N_s > 0$ will correspond to $T_1 > T_2$ and $N_s < 0$ will correspond to $T_2 > T_1$. Considering now the representations (23*a*) and (23*b*), it is seen that the shapes of the temperature profiles for $N_s > 0$ and $N_s < 0$ are the same, provided that the former is plotted as a function of Z and the latter is plotted as a function of $Z' = 1-Z$.

† A study by A. W. Goldstein (1959) involving unsteady heating conditions and time-dependent body forces also deals with a non-linear initial temperature profile.

Before going on to the actual stability computation, it is worth while considering the imposed temperature distributions and their implications in somewhat greater detail. A graphical presentation of the temperature distribution is made in figure 2. The left-hand ordinate and lower abscissa correspond to the situation wherein the temperature of the lower bounding surface is higher than that of the upper bounding surface, i.e. $N_s > 0$. The right-hand ordinate and upper abscissa correspond to the situation in which the temperature of the lower surface is less than that of the upper surface, $N_s < 0$.

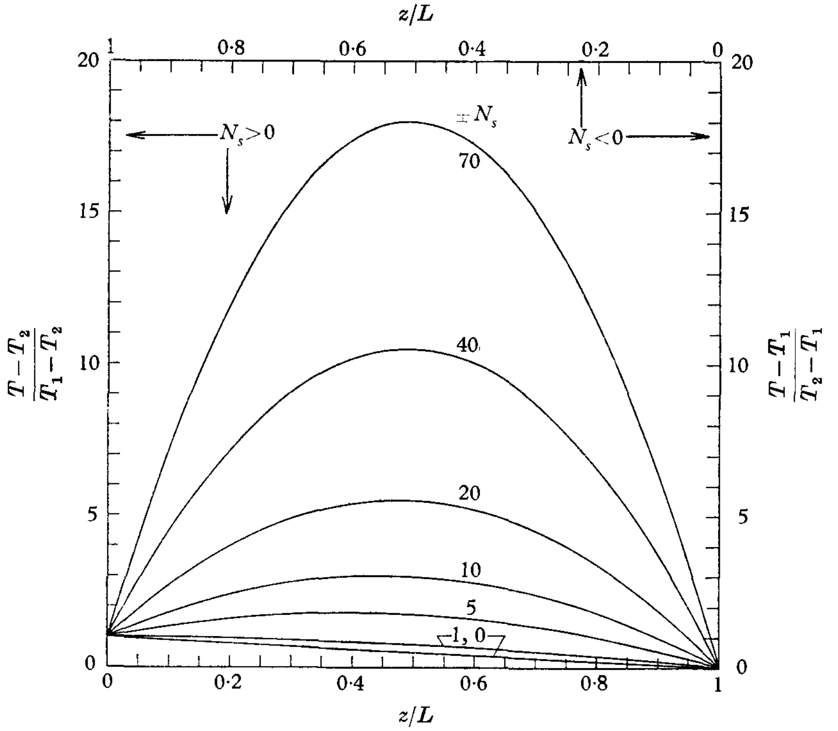


FIGURE 2. Temperature profiles in a quiescent fluid layer resulting from internal heat generation.

Consider first the case in which $N_s > 0$. In the range $0 \leqslant N_s \leqslant 1$, the highest temperature in the fluid layer occurs at the lower bounding surface, $Z = 0$. Correspondingly, one would not expect the stability characteristics of such a layer to be very different from that for the case of the linearly decreasing temperature. As N_s increases beyond unity, temperatures in excess of that at $Z = 0$ occur within the fluid. Further increases in N_s give rise to corresponding increases in fluid temperature, and the location of the temperature maximum approaches closer to $Z = \frac{1}{2}$.

The situation is somewhat different for $N_s < 0$. In the range $-1 \leqslant N_s < 0$, the temperature is monotonically increasing with height and the quiescent state is completely stable. However, for $N_s < -1$, temperatures within the fluid exceed that of the upper bounding surface, with the consequence that heavier fluid lies above lighter fluid and instability becomes possible.

4.2. *Stability computation and results*

The actual computation of the stability criteria is carried out in a manner similar to that already described in the study relating to the various boundary conditions. The series solution (11) is utilized as before, but now the parameters Λ and Ω which appear in the recursion relation (9*a*) become

$$\Lambda = a^2\Re(1-N_s)-a^6, \quad \Omega = 2a^2\Re N_s. \tag{25}$$

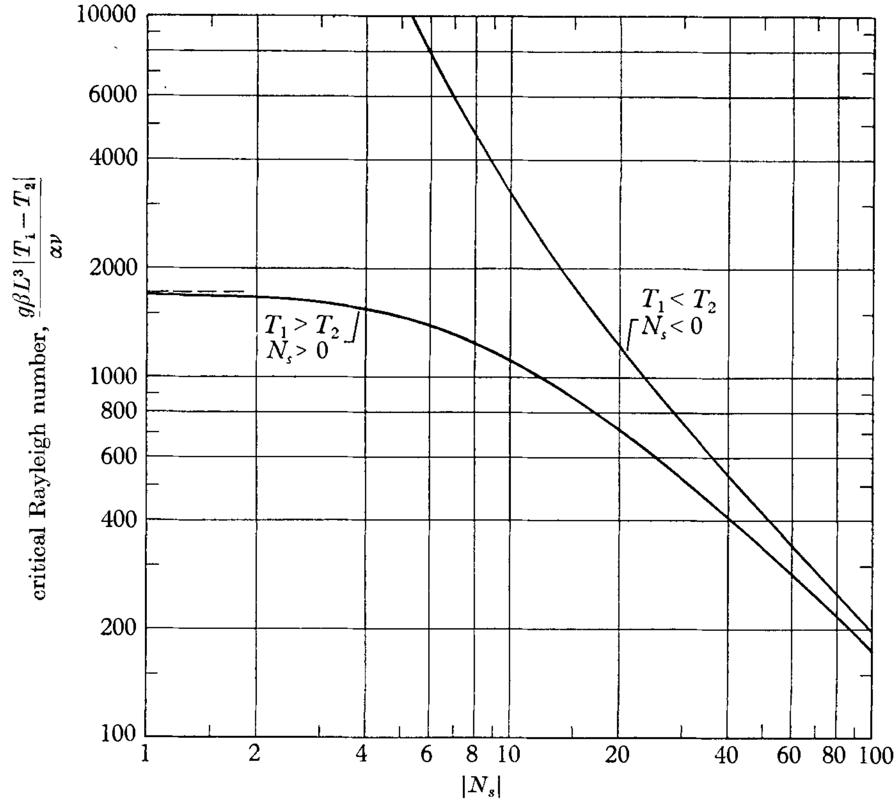


FIGURE 3. Critical Rayleigh numbers corresponding to non-linear initial temperature profiles and rigid, isothermal bounding surfaces.

Corresponding to the four constants c_2, c_3, c_4 and c_5, one may derive four equations by requiring that equation (11) satisfy condition (16*a*) at $Z = 0$ and conditions (16*a*) and (10) at $Z = 1$. The equations thus derived are linear and homogeneous, and a solution is possible only when the determinant of the coefficients vanishes. The value of the determinant depends on the parameters N_s, $\Re$ and a. It is found that for a given value of N_s, there is a minimum value of the Rayleigh number† below which the determinant cannot be zero. This, therefore, corresponds to the critical Rayleigh number marking the onset of instability.

The critical Rayleigh numbers which have been computed in this way are plotted in figure 3 for parametric values of N_s and are also listed in table 2.

† In this context, the temperature difference appearing in the Rayleigh number is $|T_1 - T_2|$.

The figure contains two curves, one corresponding to the case in which $T_1 > T_2$ ($N_s > 0$) and the second corresponding to $T_1 < T_2$ ($N_s < 0$). Considering the first of these curves, it is seen that the stability criteria for layers characterized by $N_s < 1$ are very little different from that of the linear temperature case which is indicated on the figure as a dashed horizontal line. As N_s increases, the critical Rayleigh number decreases, at first rather slowly and later quite rapidly. The non-linear temperature distribution is clearly a destabilizing influence.

The second curve on figure 3 ($T_1 < T_2$) displays some interesting differences from that described above. First of all, as N_s approaches minus one, the curve approaches infinity (complete stability at any finite value of Rayleigh number). With increases in $|N_s|$, the Rayleigh number drops sharply. At large values of $|N_s|$, there is little difference between the stability criteria for $T_1 > T_2$ and $T_1 < T_2$; this is quite reasonable, since the actual temperature profiles (i.e. T versus Z) are very little different.

	For $N_s \geqslant 0$			For $N_s < 0$		
$\pm N_s$	a	$\mathfrak{R}$	$\tilde{\mathfrak{R}}$	a	$\mathfrak{R}$	$\tilde{\mathfrak{R}}$
0	3·12	1,707·765	1,707·765	—	—	—
0·1	3·12	1,707·636	1,707·636	—	—	—
0·25	3·12	1,706·953	1,706·953	—	—	—
0·5	3·12	1,704·453	1,704·453	—	—	—
1	3·13	1,694·953	1,694·953	—	—	—
1·5	3·14	1,679·407	1,012·374	—	—	—
2·5	3·18	1,632·886	686·098	—	—	—
3	—	—	—	6·13	47,673·615	588·563
5	3·30	1,462·863	568·761	5·10	11,527·500	590·208
7·5	3·43	1,279·267	560·610	4·73	5,172·813	592·787
10	3·53	1,118·430	562·888	4·59	3,215·221	593·299
15	3·68	878·339	568·522	4·38	1,783·818	592·210
20	3·74	717·201	572·094	4·28	1,221·732	590·846
30	3·82	521·403	575·900	4·18	744·170	588·880
40	3·86	408·558	577·807	4·14	533·579	587·668
70	3·92	247·075	580·201	4·08	287·819	585·899
100	3·94	176·936	581·130	4·06	196·891	585·129
∞	4·00	—	583·206	4·00	—	583·206

TABLE 2. Critical Rayleigh numbers for non-linear temperature distribution.

Table 2 provides the detailed information from which figure 3 was prepared and additionally contains the a values corresponding to these critical Rayleigh numbers. There is also listed in table 2 a parameter $\tilde{\mathfrak{R}}$. The latter is a Rayleigh number constituted as follows: the temperature difference is formed between the maximum temperature in the fluid T_m and the temperature of the upper surface T_2. The length dimension is the distance L_m between the temperature maximum and the upper surface. Thus,

$$\tilde{\mathfrak{R}} = g\beta(T_m - T_2)\, L_m^3/\alpha\nu. \tag{26}$$

When N_s is positive and $\leqslant 1$, $\tilde{\mathfrak{R}} = \mathfrak{R}$. For other values of N_s, $\tilde{\mathfrak{R}}$ is readily computed from $\mathfrak{R}$ in conjunction with $(T_m - T_2)$ and L_m from equation (23a). From table 2, it is seen that for all N_s extending from 4† to ∞, $\tilde{\mathfrak{R}}$ falls in the surprisingly narrow range between 560 and 595. This is true regardless of whether N_s is positive or negative. From this, one might conjecture that when the location of the maximum fluid temperature is sufficiently far removed from the lower bounding surface, then the presence of this surface does not influence the stability of the layer above it.

To complete the presentation of results, a word may be said about the case $N_s = \infty$. This corresponds to a situation in which $T_1 = T_2$; consequently, a Rayleigh number based on the temperature difference $(T_1 - T_2)$ does not convey any information about the stability characteristics. However, the $\tilde{\mathfrak{R}}$ continues to provide a sufficient representation of the stability result, where it is noted that $T_m - T_2 = SL^2/8k$ and $L_m = 0{\cdot}5L$.

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† $\tilde{\mathfrak{R}}$ for $N_s = 4$ was computed from the $\mathfrak{R}$ value read from figure 3.